16. Triple Integrals Part 1

In this lecture, we will discuss

- Definition of the triple integrals
    - over a rectangular box
    - over a general bounded region E
- Three types of the bounded region
- Triple Integrals in Cylindrical and Spherical Coordinates, Part 1

Definition of the triple integrals

Triple integral of f over the box B

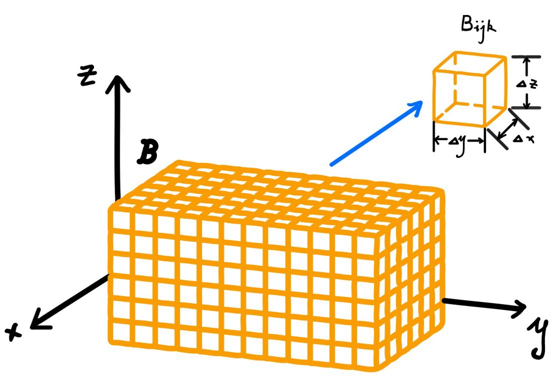
We first consider the simplest case where f is defined on a rectangular box:

$$B = \{(x, y, z) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d, r \leqslant z \leqslant s\} \tag{1}$$

- Divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz.
- The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into $l \times m \times n$ sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

  shown in the figure below.



- Note each sub-box has volume $\Delta V = \Delta x \Delta y \Delta z$.
- Similar to the case of double integral, we form the triple Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*}\right)\Delta V$$

where the sample point $\left(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*}\right)$ is in B_{ijk}.

- We choose it to be the sample point (x_i, y_j, z_k) we get a simpler-looking expression below

Definition The triple integral of f over the box B is

$$\iiint_B f(x,y,z)dV = \lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i, y_j, z_k)\Delta V$$

if this limit exists.

Similar to the double integral, we have

Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = [a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)dV = \int_r^s\int_c^d\int_a^b f(x,y,z)dxdydz$$

- The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z.
- There are five other possible orders in which we can integrate, all of which give the same value.
- For example, if we integrate with respect to y, then z, and then x, we have

$$\iiint_B f(x,y,z)dV = \int_a^b\int_r^s\int_c^d f(x,y,z)dydzdx$$

Triple integral of f over a general bounded region E

- Again similar to the case of double integral, we put E in a box B of the type given by Equation (1) .
- Then we can define a function F such that it agrees with f on E but is 0 for points in B that are outside E .
- Thus we have

$$\iiint_E f(x,y,z)dV = \iiint_B F(x,y,z)dV$$

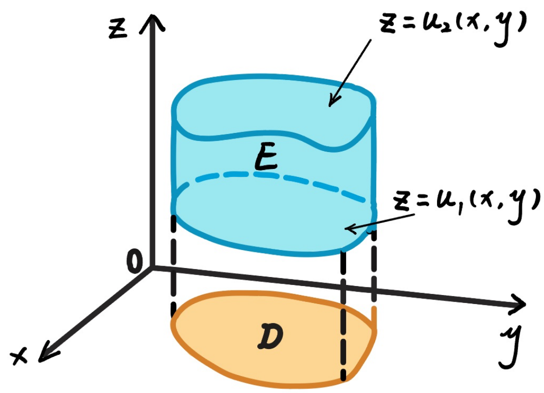
Three types of the bounded region

Type 1 Region

A solid region E is said to be of type 1 if it lies between the graphs of two continuous functions of x and y,

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leqslant z \leqslant u_2(x, y)\}, \tag{2}$$

where D is the projection of E onto the xy-plane as shown in the figure below.



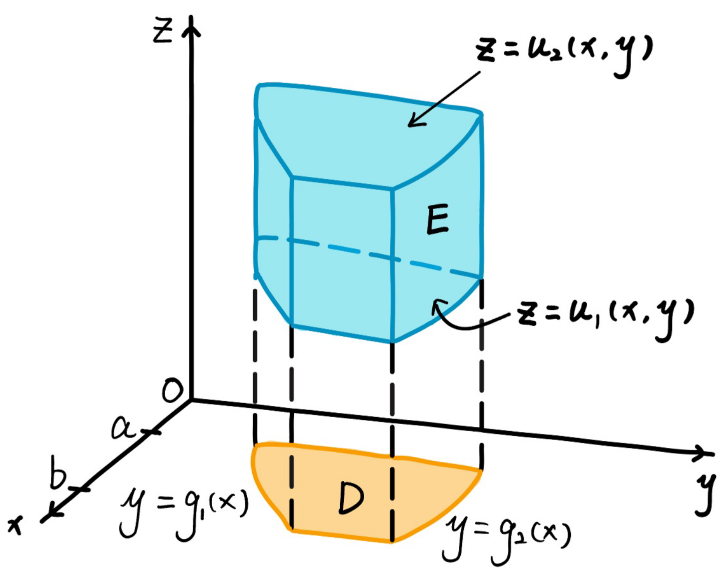
A Type 1 Region

Similar to the case of double integral, it can be shown that if E is a type 1 region given by Equation (2), then

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz\right]dA \tag{3}$$

Particularly, if the projection D of E onto the xy-plane is a type 1 plane region (see Lecture 14) as in the figure below, then

$$E = \{(x, y, z) \mid a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x), u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$



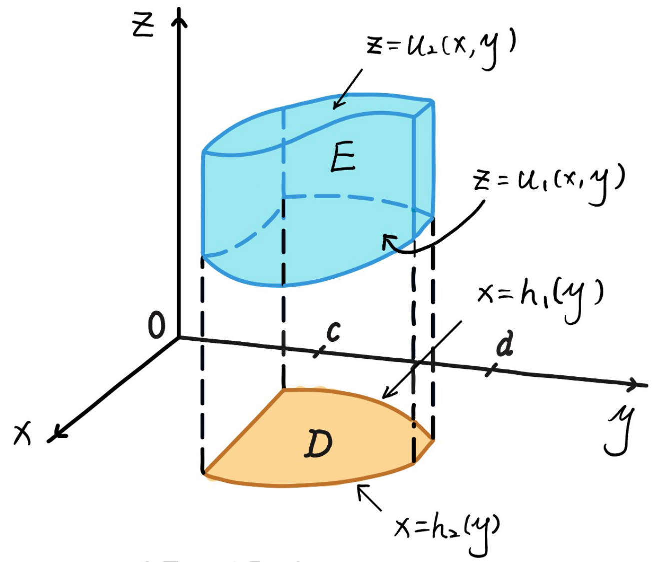
A Type 1 Region

Therefore, Equation (2) becomes

$$\iiint_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdydx \tag{4}$$

On the other hand, if D is a type 2 plane region (see Lecture 14) as in the following figure, then

$$E = \{(x,y,z) \mid c \leqslant y \leqslant d, h_1(y) \leqslant x \leqslant h_2(y), u_1(x,y) \leqslant z \leqslant u_2(x,y)\}$$



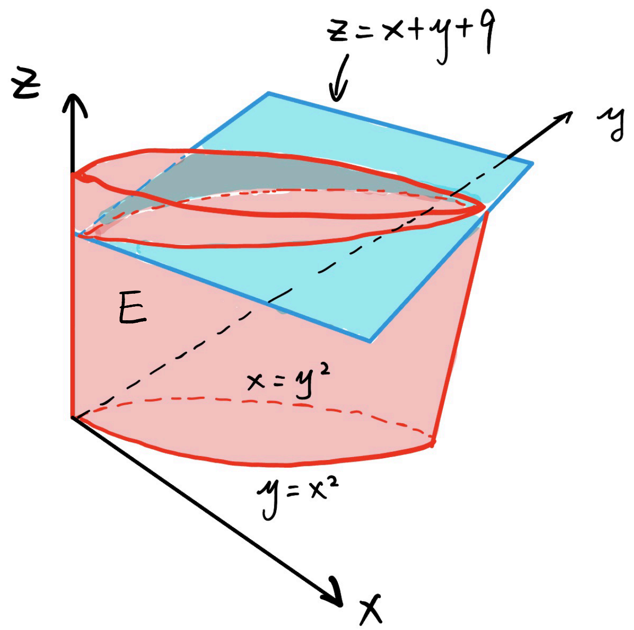
A Type 1 Region

Then, Equation (2) becomes

$$\iiint_E f(x,y,z)dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdxdy \tag{5}$$

$Vol(E) = \iiint_E 1\, dV$

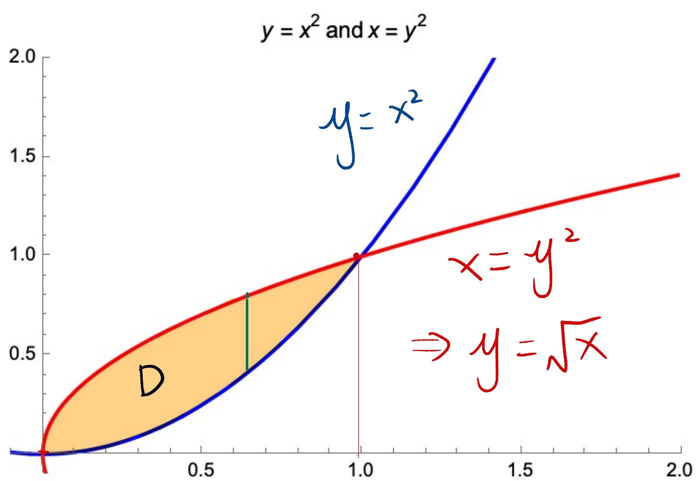
Example 1. Find the volume of the solid in $\mathbf{R}^3$ bounded by $y = x^2, x = y^2, z = x + y + 9$, an $z = 0$.



ANS: The solid E is shown in the left figure.

$0 \le z \le x+y+9$

The projection of E onto the xy plane is the region in the 1st quadrant enclosed by curves $y=x^2$ and $x=y^2$



Note the volumn of E is

$$Vol(E) = \iiint_E 1\, dV$$

$$= \iint_D \int_0^{x+y+9} 1\, dz\, dA$$

$$= \iint_D z \Big|_{z=0}^{z=x+y+9} dA$$

$$= \iint_D x+y+9\, dA$$

(Now this is the usual double integral question over D covered in Lecture 14 & 15)

Thus $\iint_D x+y+9\, dA$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} x+y+9\, dy\, dx$$

$$= \int_0^1 xy + \frac{1}{2}y^2 + 9y \Big|_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left(x\sqrt{x} + \frac{1}{2}x + 9\sqrt{x}\right) - \left(x^3 + \frac{1}{2}x^4 + 9x^2\right) dx$$

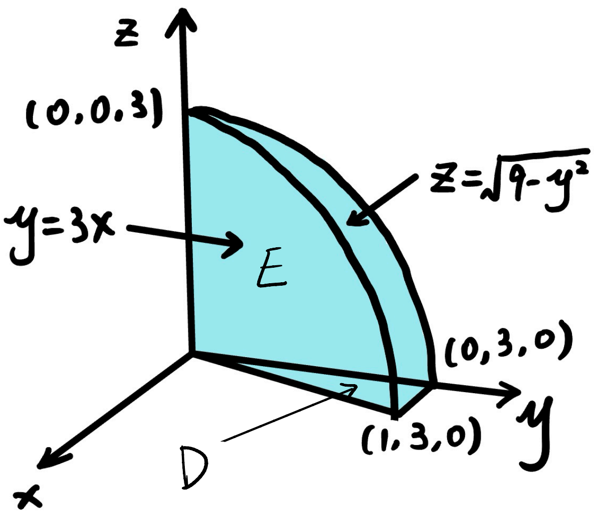
$$= \int_0^1 -\frac{1}{2}x^4 - x^3 - 9x^2 + x^{\frac{3}{2}} + \frac{1}{2}x + 9x^{\frac{1}{2}} \, dx$$

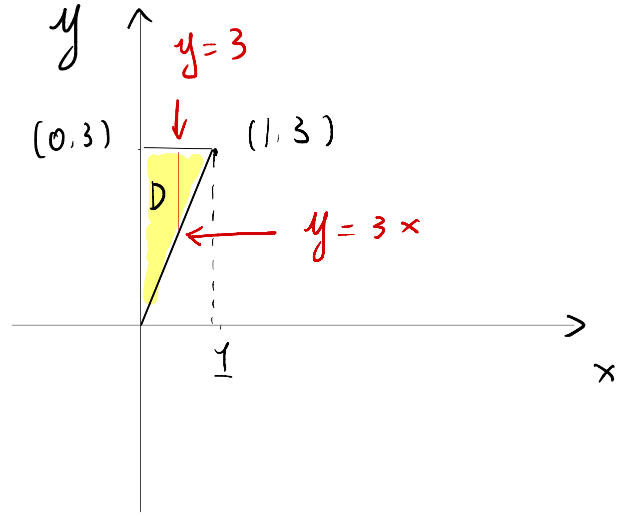
$$= -\frac{1}{2}\cdot\frac{1}{5}x^5 - \frac{1}{4}x^4 - 3x^3 + \frac{1}{1+\frac{3}{2}}x^{\frac{3}{2}+1} + \frac{1}{2}\cdot\frac{1}{2}x^2 + \frac{9}{1+\frac{1}{2}}x^{\frac{3}{2}} \Big|_0^1$$

$$= -\frac{1}{10} - \frac{1}{4} - 3 + \frac{1}{4} + \frac{2}{5} + 6$$

$$= \frac{66}{20}$$

Example 2. Evaluate the triple integral $\iiint_E z dV$, where E is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0, y = 3x$, and $z = 0$ in the first octant.





ANS: $\iiint_E z\,dV = \iint_D \int_0^{\sqrt{9-y^2}} z\,dz\,dA = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx$

$$= \int_0^1 \int_{3x}^3 \frac{1}{2} z^2 \Big|_{z=0}^{z=\sqrt{9-y^2}} dy\,dx = \int_0^1 \int_{3x}^3 \frac{1}{2}(9-y^2)\,dy\,dx$$

$$= \int_0^1 \left[\frac{9}{2} y - \frac{1}{2}\cdot\frac{1}{3} y^3\right]_{y=3x}^{y=3} dx$$

$$= \int_0^1 \frac{9}{2}\cdot 3 - \frac{1}{6}\cdot 3^3 - \left(\frac{9}{2}\cdot 3x - \frac{1}{6}\cdot(3x^3)\right) dx$$

$$= \int_0^1 \left(9 - \frac{27}{2} x + \frac{9}{2} x^3\right) dx$$

$$= 9x - \frac{27}{4} x^2 + \frac{9}{8} x^4 \Big|_0^1$$

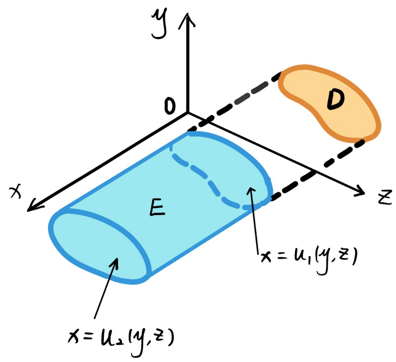
$$= \frac{27}{8}$$

Type 2 region

A solid region E is of type 2 if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leqslant x \leqslant u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz-plane in the figure below.



A Type 2 Region

The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

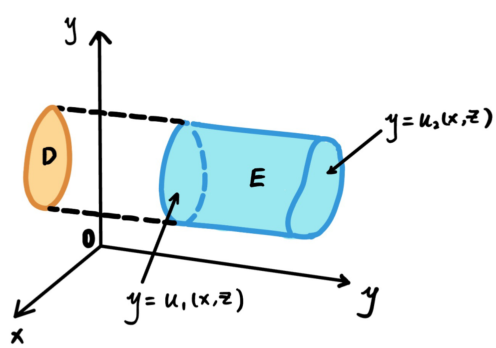
$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)dx\right]dA \tag{6}$$

Type 3 region

A type 3 region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leqslant y \leqslant u_2(x, z)\}$$

where D is the projection of E onto the xz-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface in the figure below.



A Type 3 Region

For this type of region we have

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)dy\right]dA \tag{7}$$

Remark: Each of Equations (6) and (7) may have two possible expressions for the integral depending on whether D is a type 1 or type 2 plane region (in Lecture 14) corresponding to Equations (4) and (5).

Example 3. Rewrite the triple integral $\int_0^1 \int_0^x \int_0^y f(x,y,z)dzdydx$ as $\int_a^b \int_{g_1(z)}^{g_2(z)} \int_{h_1(y,z)}^{h_2(y,z)} f(x,y,z)dxdydz$

ANS: Denote the integral region by E.

From the given format, we know

$$0 \le z \le y$$
$$0 \le y \le x$$
$$0 \le x \le 1$$

Thus we have $0 \le z \le y \le x \le 1$ ⊛

To get the format of $\int_a^b \int_{g_1(z)}^{g_2(z)} \int_{h_1(y,z)}^{h_2(y,z)} f(x,y,z)dxdydz$

we see from ⊛.

$$y \le x \le 1$$
$$z \le y \le 1$$
$$0 \le z \le 1$$

Thus $a = 0$, $b = 1$, $g_1(z) = z$, $g_2(z) = 1$,

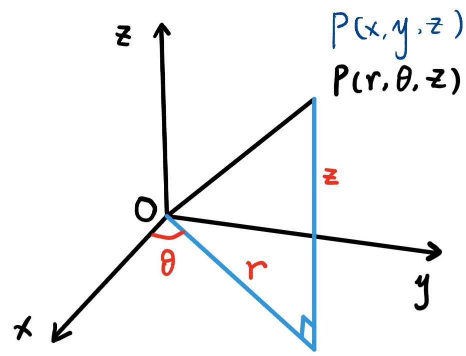
$h_1(y,z) = y$, $h_2(y,z) = 1$.

Triple Integrals in Cylindrical and Spherical Coordinates, Part 1

In this section, we discuss that some triple integrals are easier to evaluate using cylindrical or spherical coordinates.

We will talk about the cylindrical coordinates in part 1. The spherical coorinates will be covered in Lecture 17.

- Recall from Lecture 7 that the cylindrical coordinates of a point P are (r, θ, z), where r, θ, and z are shown in the figure below.
- Suppose that E is a type 1 region whose projection D on the xy-plane is described in polar coordinates in the figure below.

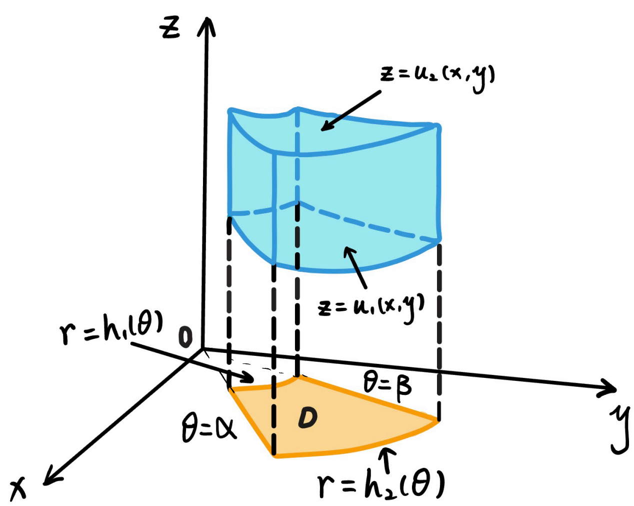


- Suppose that f is continuous and

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$



- From Equation (3) we have

$$\iiint_E f(x, y, z) dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) dz\right] dA \tag{8}$$

If f is continuous on a polar region of the form $D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$, then

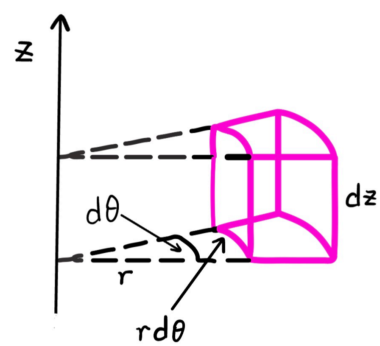
$\iint_D f(x, y) dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta.$

- We also know how to evaluate double integrals in polar coordinates. Combining Equation (8) with the equation in Lecture 15 on page 12, we obtain the following formula

Theorem 1 Formula for triple integration in cylindrical coordinates

$$\iiint_E f(x, y, z) dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta \qquad (9)$$

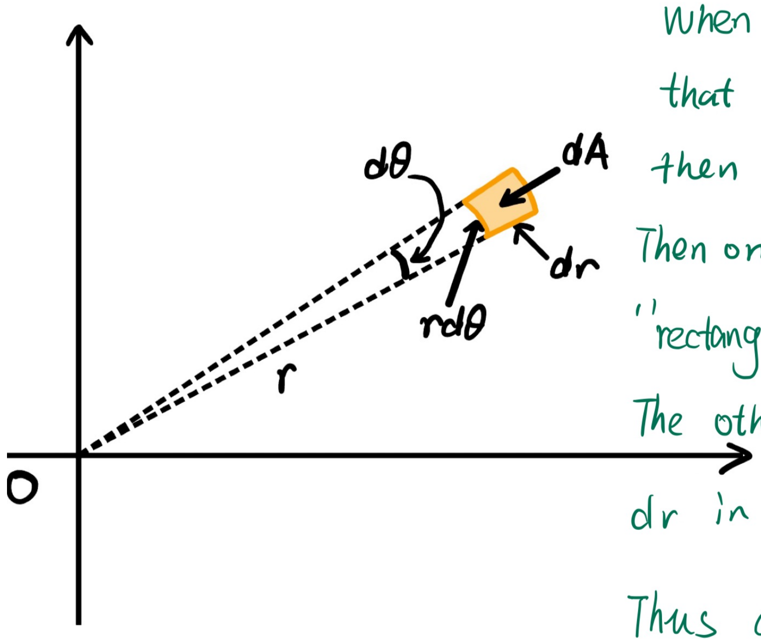
- The formula states that we can convert a triple integral from rectangular to cylindrical coordinates by writing $x = r\cos\theta, y = r\sin\theta$, leaving z as it is, using the appropriate limits of integration for z, r, and θ, and replacing dV by $r dz dr d\theta$.
- The figure below shows how to remember the relationship $dV = r dz dr d\theta$



$dV = r dz dr d\theta$

Recall we explained why the bottom "rectangle" has area $r dr d\theta$ in lecture 15.

Thus dV is obtained by multiplying the area $r dr d\theta$ with the height dz.



When θ is small enough, that is, when $\theta \approx 0$, then $\theta \approx \sin\theta$

Then one side of the "rectangle" has length $\approx r d\theta$

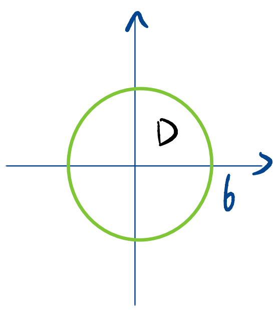
The other side has length dr in an obvious way.

Thus $dA = r dr d\theta$.

Example 4. Use cylindrical coordinates to calculate :

$$\iiint_{\mathcal{W}} x^2 + y^2 dV \quad \mathcal{W} : x^2 + y^2 \leq 36, \quad 0 \leq z \leq 12$$

ANS: The projection of W onto the xy-plane is the region inside the circle $x^2+y^2=36$.



In polar coordinates,

$D: \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 6$

The upper and lower boundries are planes $z=0$ and $z=12$.

Thus, the region W has the following description in cylindracal coordinates:

$W: 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 6, \quad 0 \leq z \leq 12$

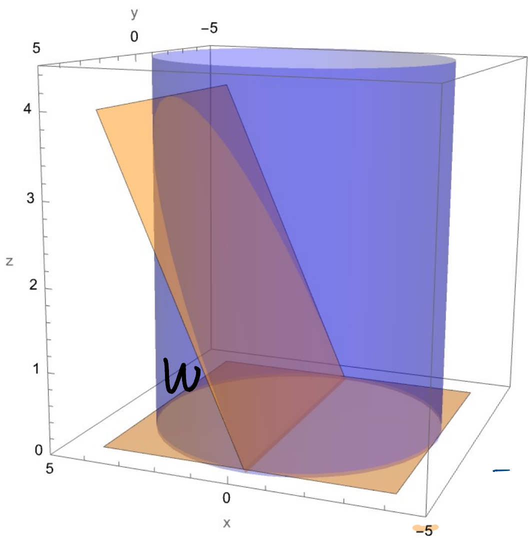
Therefore

$$\iiint_W (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^6\int_0^{12} \left(r^2\cos^2\theta + r^2\sin^2\theta\right) r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^6\int_0^{12} r^3\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^6 r^3 z\Big|_{z=0}^{z=12} dr\,d\theta$$

$$= 12\int_0^{2\pi} \frac{1}{4} r^4\Big|_0^6 d\theta = 3\cdot 6^4\,\theta\Big|_0^{2\pi} = 6\cdot 6^4\pi = 6^5\pi$$

Example 5. Find the volume of the wedge-shaped region contained in the cylinder $x^2 + y^2 = 16$ and bounded above by the plane $z = x$ and below by the xy-plane.

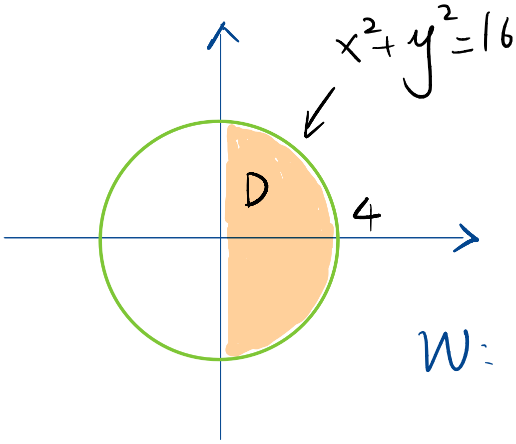


ANS: Similar to the previous problem, we start from expressing the given region, say W, in terms of polar coordinates

- W is bounded above by the plane $z = x$ and below by $z = 0$. Thus

$$0 \le z \le x = r\cos\theta \quad ⊛$$

- Particularly, $x \ge 0$ from ⊛. Thus the projection of W onto xy-plane is the following semicircle:



Thus

$$D: -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 4$$

Therefore:

$$W: 0 \le z \le r\cos\theta, \; -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 4$$

Thus the volumn of the solid in terms of the cylindrical coordinates is

$$\iiint_W 1\,dV = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^4 \int_0^{r\cos\theta} 1 \; r\,dz\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^4 rz \Big|_{z=0}^{z=r\cos\theta} dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^4 r^2\cos\theta \; dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3} r^3 \cos\theta \Big|_0^4 d\theta$$

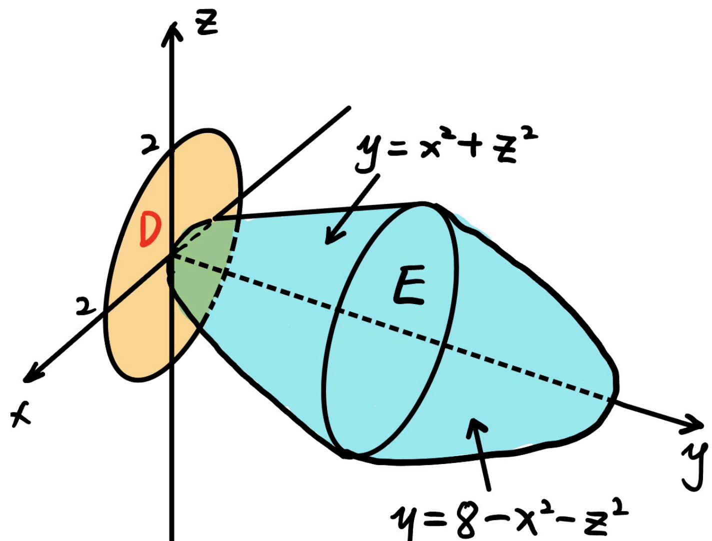
$$= \frac{4^3}{3} \sin\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{4^3}{3} \left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{2 \cdot 4^3}{3}$$

Exercise 6. Use a triple integral to find the volume of the solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$.

Answer.



The paraboloids intersect when $x^2 + z^2 = 8 - x^2 - z^2$, which implies $x^2 + z^2 = 4$.

Therefore the intersection is the circle $x^2 + z^2 = 4$, $y = 4$.

The projection of E onto the xz-plane is the disk $x^2 + z^2 \leq 4$, so

$$E = \left\{(x, y, z) \mid x^2 + z^2 \leq y \leq 8 - x^2 - z^2, x^2 + z^2 \leq 4\right\}.$$

Let $D = \left\{(x, z) \mid x^2 + z^2 \leq 4\right\}$.

Using polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$, we have

$$\begin{aligned} V &= \iiint_E dV = \iint_D \left(\int_{x^2+z^2}^{8-z^2} dy\right) dA = \iint_D (8 - 2x^2 - 2z^2)\, dA \\ &= \int_0^{2\pi} \int_0^2 (8 - 2r^2) r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 (8r - 2r^3)\, dr \\ &= [\theta]_0^{2\pi} \left[4r^2 - \frac{1}{2} r^4\right]_0^2 = 2\pi(16 - 8) = 16\pi \end{aligned}$$

Exercise 7. Integrate $f(x,y,z)=20xz$ over the region in the first octant $(x,y,z\geq 0)$ above the parabolic cylinder $z=y^2$ and below the paraboloid $z=8-2x^2-y^2$.

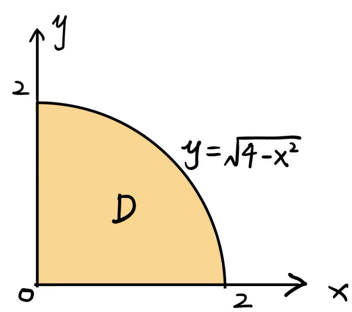
Answer.

We first find the projection of the region E onto the xy-plane.

We find the curve of intersection between the upper and lower surfaces, by solving the following equation for $x,y\geq 0$:

$$8-2x^2-y^2=y^2 \quad\Rightarrow\quad y^2=4-x^2 \quad\Rightarrow\quad y=\sqrt{4-x^2}, x\geq 0$$

The projection D of E onto the xy-plane is the region bounded by the circle $x^2+y^2=4$ and the positive axes.



We compute the triple integral over E by evaluating the following iterated integral:

$$\begin{aligned}\iiint_E 20xz\,dV &= \iint_D \left(\int_{y^2}^{8-2x^2-y^2} 20xz\,dz\right)dA\\ &= \iint_D 10xz^2\Big|_{z=y^2}^{8-2x^2-y^2} dA\\ &= \iint_D 10x\left(\left(8-2x^2-y^2\right)^2-y^4\right)dA\\ &= \iint_D 10x\left(8-2x^2-2y^2\right)\left(8-2x^2\right)dA\\ &= \iint_D 40x\left(4-x^2\right)\left(4-x^2-y^2\right)dA\end{aligned}$$

$$=\int_0^2\left(40x\left(4-x^2\right)\int_0^{\sqrt{4-x^2}}\left(4-x^2-y^2\right)dy\right)dx$$

$$=\int_0^2 40x\left(4-x^2\right)\left(\left(4-x^2\right)y-\frac{y^3}{3}\right)\Big|_{y=0}^{\sqrt{4-x^2}}dx$$

$$=\int_0^2 40x\left(4-x^2\right)\left(\left(4-x^2\right)^{3/2}-\frac{\left(4-x^2\right)^{3/2}}{3}\right)dx=\frac{80}{3}\int_0^2\left(4-x^2\right)^{5/2}x\,dx$$

To find the antiderivative of $\int\left(4-x^2\right)^{5/2}xdx$, we have

$$\int\left(4-x^2\right)^{5/2}xdx=-\frac{1}{2}\int(4-x^2)^{5/2}d(4-x^2)=-\frac{1}{7}\left(4-x^2\right)^{7/2}$$

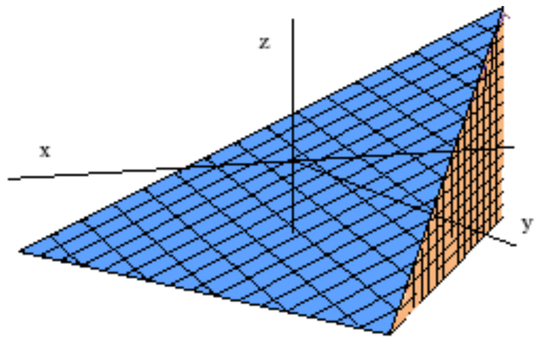
Thus

$$\begin{aligned}\iiint_E 20xzdV &= \frac{80}{3}\int_0^2\left(4-x^2\right)^{5/2}xdx\\ &= \frac{80}{3}\cdot(-\frac{1}{7})\left(4-x^2\right)^{7/2}|_0^2\\ &= \frac{80}{21}\cdot 2^7=\frac{10240}{21}\approx 487.619\end{aligned}$$

Exercise 8. Find the volume of the pyramid with base in the plane $z = -10$ and sides formed by the three planes $y = 0$ and $y - x = 3$ and $x + 2y + z = 3$.

Answer.

The pyramid is shown in the figure below.



The planes $y = 0$, and $y - x = 3$, and $x + 2y + z = 3$ intersect the plane $z = -10$ in the lines $y = 0, y - x = 3, x + 2y = 13$ on the $z = -10$ plane (think about looking straight down on the pyramid).

Let R be the triangle in the planes $z = -10$ with the above three points as vertices. Then, the volume of the solid is

$$V = \int_0^{16/3} \int_{y-3}^{(13-2y)} \int_{-10}^{3-x-2y} dz dx dy = \int_0^{16/3} \int_{y-3}^{(13-2y)} 13 - x - 2y dx dy$$

Integrating with respect to x, we have

$$V = \int_0^{16/3} \left((13 - 2y)x - \frac{1}{2}x^2 \right) \Bigg|_{x=y-3}^{x=(13-2y)} dy$$

so that

$$V = \int_0^{16/3} (13 - 2y)(16 - 3y) - \frac{1}{2}\left((13 - 2y)^2 - (y - 3)^2\right) dy.$$

To evaluate this last integral we expand the squared terms and then proceed as expected, getting

$$V = \frac{2048}{9}$$

Exercise 9. Use cylindrical coordinates to calculate $\iiint_{\mathcal{W}} f(x, y, z) dV$ for the given function and region:

$$f(x, y, z)=z, \quad x^{2}+y^{2} \leq z \leq 25$$

Answer.

The upper boundary of $\mathcal{W}$ is the plane $z=25$, and the lower boundary is $z=x^{2}+y^{2}=r^{2}$. Therefore, $r^{2} \leq z \leq 25$.

The projection $\mathcal{D}$ onto the xy-plane is the circle $x^{2}+y^{2}=25$ or $r=5$. That is,

$$\mathcal{D}: 0 \leq \theta \leq 2 \pi, 0 \leq r \leq 5$$

The inequalities defining $\mathcal{W}$ in cylindrical coordinates are thus

$$\mathcal{W}: 0 \leq \theta \leq 2 \pi, 0 \leq r \leq 5, r^{2} \leq z \leq 25$$

Thus we have the following integral:

$$\iiint_{\mathcal{W}} z dV=\int_{0}^{2 \pi} \int_{0}^{5} \int_{r^{2}}^{25} z r dz dr d\theta=\int_{0}^{2 \pi} \int_{0}^{5} \left.\frac{z^{2} r}{2}\right|_{z=r^{2}} ^{25} dr d\theta=$$
$$\int_{0}^{2 \pi} \int_{0}^{5} \frac{r\left(625-r^{4}\right)}{2} dr d\theta=\int_{0}^{2 \pi} \int_{0}^{5} \frac{625 r-r^{5}}{2} dr d\theta=\int_{0}^{2 \pi} \left.\frac{625 r^{2}}{4}-\frac{r^{6}}{12}\right|_{0} ^{5} d\theta$$
$$=2604.17 \int_{0}^{2 \pi} d\theta=5208.33 \pi$$

Exercise 10. Evaluate the triple integral of $f(x, y, z)=\cos \left(x^{2}+y^{2}\right)$ over the solid cylinder with height 8 and with base of radius 2 centered on the z axis at $z=-3$.

Answer.

By Equation (9) in Theorem 1, we have

$$\int_{W} f dV=\int_{-3}^{5} \int_{0}^{2 \pi} \int_{0}^{2} \cos \left(r^{2}\right) r dr d\theta dz$$

Integrating,

$$\int_{-3}^{5} \int_{0}^{2 \pi} \int_{0}^{2} \cos \left(r^{2}\right) r dr d\theta dz=\int_{-3}^{5} \int_{0}^{2 \pi} \left.\frac{1}{2} \cdot \sin \left(r^{2}\right)\right|_{r=0} ^{r=2} d\theta dz$$
$$=\int_{-3}^{5} \int_{0}^{2 \pi} \frac{1}{2}(\sin (4)) d\theta dz=\frac{1}{2}(\sin (4)) \int_{-3}^{5} 2 \pi dz=8 \pi(\sin (4))$$